

Asymptotics and Renewal Approximation in the Online Selection of Increasing Subsequence

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We revisit the problem of maximising the expected length of increasing subsequence that can be selected from a marked Poisson process by an online strategy. Resorting to a natural size variable, we represent the problem in terms of a controlled piecewise deterministic Markov process with decreasing paths. We apply a comparison method to the optimality equation to obtain fairly complete asymptotic expansions for the moments of the maximal length, and, with the aid of a renewal approximation, give a novel proof to the central limit theorem for the length of selected subsequence under either the optimal strategy or a strategy sufficiently close to optimality.

Keywords: Online selection, Monotone subsequence, Renewal approximation, Dynamic programming.

1. Introduction

In the stochastic optimisation problem of Samuels and Steele [25], a sequence of independent random marks with known continuous distribution is observed at occurrences of the unit-rate Poisson process with horizon t . Every time a mark is observed, it can be selected or rejected, with every decision becoming immediately final. The selected subsequence must increase. The objective is to maximise the expected length of subsequence chosen in the *online* fashion, that is using a nonanticipating decision strategy.

A prophet with complete foresight of the sequence could adopt an *offline* algorithm to select one of the longest possible increasing subsequences, of some maximal length $l(t)$. The question about the asymptotic behaviour of $l(t)$ for large t is the well-known Ulam-Hammersley problem. A central achievement here is the work by Baik, Deift and Johansson [5] where it was shown that $(l(t) - 2\sqrt{t})/t^{1/6}$ converges in distribution to the Tracy-Widom law from the random matrix theory, whereas the expected length has asymptotics

$$\mathbb{E}l(t) = 2\sqrt{t} + ct^{1/6} + o(t^{1/6}), \quad \text{as } t \rightarrow \infty$$

with $\hat{c} \doteq -1.758$. Historically, finding just the leading term of the expansion stimulated considerable development. See Romik's book [24] for a detailed account.

Without clairvoyance each feasible decision incurs some kind of risk. A rejected mark sustains a missed opportunity which might not be compensated in the future, while an accepted mark reduces the range of subsequent choices that must meet the monotonicity constraint.

Let $L(t)$ be the length of selected subsequence under the optimal online strategy. The *value function* $v(t) := \mathbb{E}L(t)$ satisfies an integro-differential optimality equation, which does not seem to admit a closed-form solution. Samuels and Steele [25] found the leading asymptotics $v(t) \sim \sqrt{2t}$, where the order was identified by Hammersley's subadditivity method. Bruss and Delbaen [8] combined a thorough analysis of the optimality equation with martingale methods to derive much tighter estimates

$$\sqrt{2t} - \log(1 + \sqrt{2t}) + c < v(t) < \sqrt{2t}, \quad (1)$$

(with explicit c) and to show that similar bounds hold for the variance $\text{Var} L(t)$. In another paper Bruss and Delbaen [9] extended this technique to obtain a functional limit theorem for fluctuations of the shape of selected subsequence, showing in particular that the distribution of $\sqrt{3} (L(t) - \sqrt{2t})/(2t)^{1/4}$ converges to normal. Note that the second term in approximation (1) to the mean and the leading asymptotics of the variance both have the *orders* of magnitude different from that of their offline counterparts.

A parallel development occurred in the online increasing subsequence problem with a fixed number of observations n , which in fact was the main theme in [25]. Arlotto, Wei and Xie [4] introduced an adaptive selection strategy within the $O(\log n)$ gap from the optimality, thus improving upon previous estimates based on a simple stationary strategy [23, 25]. Arlotto, Nguyen and Steele [3] proved a precise analogue of the Bruss-Delbaen central limit theorem for the optimal number of choices. The setting with Poisson arrivals can be related to the fixed- n problem by allowing the length of the observed sequence to be used in decision strategies. However, despite the apparent similarity, translating results from one model to the other is not automatic since the information flows are very different. Moreover, yet another informational environment appears in the problem where observations arrive at fixed times but the horizon is a random variable with Poisson distribution of rate t [14].

The online increasing subsequence problem is distribution-free; hence the marks can be assumed uniformly distributed. In that special case the problem is equivalent to a sequential bin-packing problem, with the constraint that the sum of selected marks cannot exceed one [10, 12, 23]. This connection has been a source of upper bounds as in (1). Further variations studied in the literature include sequential selection of increasing chain from a partial order [6], selection from permutations [16, 22], discrete-time problems with random n [14, 17] and a dual problem of minimising the time needed to select an increasing subsequence of pre-specified length [2].

In this paper we prove the asymptotic expansion for the optimal expected length,

$$v(t) = \sqrt{2t} - \frac{1}{12} \log t + c^* + \frac{\sqrt{2}}{144\sqrt{t}} + O(t^{-1}), \quad t \rightarrow \infty, \quad (2)$$

with some unknown constant c^* , and also derive a similar expansion for $\text{Var} L(t)$. The analytical part of our approach is based on a comparison method which readily yields

approximate solutions of relevant functional equations up to a $O(1)$ term. But justifying convergence of the remainder and expansion beyond $O(1)$ require much more probabilistic insight. Our main novelty here is the representation of the selection problem in terms of a controlled piecewise deterministic Markov process, whose state variable is the square root of the expected number of remaining choosable observations. We show that, when the state variable is large, the process behaves similarly to an alternating renewal process. The renewal approximation enables us to explain the logarithmic term in (2) and to give an alternative proof of the normal approximation to $L(t)$. Unlike [8, 9] we do not rely on the concavity of the value function $v(t)$, rather use tools well suited to the analysis of a wider class of near-optimal strategies including a continuous-time analogue of the adaptive strategy from [4].

The comparison method was applied recently to obtain asymptotic expansions of the optimal values in the fixed- n increasing subsequence problem and the dual problem with the quickest-selection objective [26]. In [18] we applied the results of the present paper to refine the functional limit theorems of Bruss and Delbaen [9] by showing, in particular, that the normalised shape of the optimal online subsequence converges to a Brownian bridge.

Notation. We will use \sim for asymptotic expansions without explicit estimate of the remainder, e.g. $f(t) \sim f_1(t) + f_2(t) + \dots + f_k(t)$ as $t \rightarrow \infty$ means that $f_{i+1}(t) = o(f_i(t))$ for $1 \leq i < k$. We denote c^* , c_0 , c_1 , etc. some absolute constants, reserving symbol c for constant with context-dependent value.

2. Planar Poisson setup and the leading asymptotics

Standardising the distribution of marks to $[0, 1]$ -uniform leads to a natural setting of the problem with horizon t in terms of the unit-rate Poisson random measure in the rectangle $[0, t] \times [0, 1]$. The generic atom (s, x) is interpreted as mark x observed at time s , whereupon a selection/rejection decision must be made solely on the base of the allocation of atoms within $[0, s] \times [0, 1]$. A sequence $(s_1, x_1), \dots, (s_n, x_n)$ of atoms is said to be increasing if it is a chain in the partial order in two dimensions, that is $0 < s_1 < \dots < s_n$ and $0 < x_1 < \dots < x_n$. The task is to maximise the expected length of an increasing sequence over selection strategies adapted to the aforementioned information.

To solve the optimisation problem it is sufficient to consider a relatively small class of strategies defined recursively by means of some acceptance window $\psi(t, s, y)$ satisfying $0 \leq \psi(t, s, y) \leq 1 - y$ for $0 \leq s \leq t < \infty$ and $y \in [0, 1]$. The corresponding strategy selects observation $(s, x) \in [0, t] \times [0, 1]$ if and only if $0 < x - y \leq \psi(t, s, y)$, where y is the running maximum, i.e. the last (hence the highest) mark selected before time s , with the convention that $y = 0$ if no selections have been made. Note that the running maximum process and the selected chain uniquely determine one another.

The acceptance window can be regarded as a control function for the running maximum process, which is a right-continuous Markov process $Y = (Y(s), 0 \leq s \leq t)$ starting with $Y(0) = 0$, with piecewise constant paths increasing by positive jumps. At time s

in state y a transition occurs at rate $\psi(t, s, y)$, and given that Y jumps, the increment $Y(s) - Y(s-)$ is uniformly distributed on $[0, \psi(t, s, y)]$. The optimal control function corresponds to the process with the maximal expected number of jumps.

Intuitively, a large acceptance window steers Y from 0 to about 1 in just a few jumps. On the other hand, a small acceptance window makes the jumps rare, so the time resource expires before a substantial number of selections is made.

For instance, the *greedy* strategy has the largest possible acceptance window

$$\psi(t, s, y) = 1 - y.$$

The strategy selects the sequence of records [11], which has the expected length given by the exponential integral function

$$\text{Ein}(t) = \int_0^t \frac{1 - e^{-s}}{s} ds \sim \log t, \quad t \rightarrow \infty. \quad (3)$$

The greedy strategy is only optimal for $t \leq 1.345 \dots$, when the expected number of records (3) is not bigger than 1.

Next by the complexity is the family of *stationary* strategies, which have acceptance window of the form $\psi(t, s, y) = \delta(t) \wedge (1 - y)$, depending neither on the time of observation nor on the running maximum, as long as Y does not overshoot $1 - \delta(t)$. We sketch an argument showing that the stationary strategy with $\delta^*(t) = \sqrt{2/t}$ achieves the asymptotic optimality in the principal term. See [4, 6, 15, 25] for analogues in discrete-time models.

For shorthand write $\delta = \delta(t)$. Up to the first overshoot over $1 - \delta$, the running maximum Y coincides with a compound Poisson process S , characterised by the jump rate δ and the $[0, \delta]$ -uniform distribution of increments. As $\delta \rightarrow 0$ but so that $t\delta \rightarrow \infty$, the number of jumps of S that occur within horizon t is asymptotic to $t\delta$, and the number of jumps until S passes $1 - \delta$ is asymptotic to $2/\delta$. The maximum of $(t\delta) \wedge (2/\delta)$ is attained for $\delta^*(t) = \sqrt{2/t}$, which results in the expected length asymptotic to $\sqrt{2t}$. After the first selection above $1 - \delta^*(t)$ the strategy is greedy, with the expected number of choices being $O(1)$, hence not affecting the leading asymptotics. It remains to recall the upper bound in (1).

In the sequel, under the stationary strategy we shall mean this particular one with $\delta = \delta^*(t)$, that is with $\psi(t, s, y) = \sqrt{2/t} \wedge (1 - y)$. Due to the connection with compound Poisson process this strategy is easy to analyse. Indeed, let $L_0(t)$ be the length of increasing subsequence chosen by the stationary strategy. Representing $L_0(t)$ as a minimum of two independent renewal processes we have a limit

$$\sqrt{3} \frac{L_0(t) - \sqrt{2t}}{(2t)^{1/4}} \xrightarrow{d} \eta, \quad (4)$$

with $\eta = \xi_1 \wedge (\xi_2/\sqrt{3})$, where ξ_1 and ξ_2 are independent standard normal variables. Specialising the formulas for moments as found in [21], $\mathbb{E}\eta = -\sqrt{2/\pi}$, $\text{Var}(\eta) = 2 - 2/\pi$. Since $\mathbb{E}\eta \neq 0$, the optimality gap of the strategy appears to be $\sqrt{2t} - \mathbb{E}L_0(t) = O(t^{1/4})$.

Under the stationary strategy, the running maximum $Y(s)$ grows about linearly as time progresses, with transversal fluctuations about the diagonal $(s, s/t)$ being of the order of $t^{-1/4}$. See [18] for approximation to Y by a Brownian motion.

3. The optimality equation

The stationary strategy lacks the following important feature inherent to the overall optimal strategy. Given that at time s the running maximum is y , all future choosable observations belong to $[s, t] \times [y, 1]$, and their expected number equals $\tau := (t - s)(1 - y)$. Thus further selection becomes an independent subproblem, equivalent to the original problem in $[0, \tau] \times [0, 1]$ with horizon τ . This implies that it is sufficient to optimise over the class of strategies with acceptance window of the form

$$\psi(t, s, y) = (1 - y) \varphi((t - s)(1 - y)) \quad (5)$$

for some $\varphi : [0, \infty) \rightarrow [0, 1]$. Such a strategy accepts mark x at time s if and only if

$$0 < \frac{x - y}{1 - y} \leq \varphi((t - s)(1 - y)).$$

Strategies of this kind will be called *self-similar*.

Let $L_\varphi(t)$ be the length of subsequence selected by such self-similar strategy. The *value function* is defined as

$$v(t) = \sup_{\varphi} \mathbb{E} L_\varphi(t).$$

The value function is increasing and satisfies the *optimality equation*

$$v'(t) = \int_0^1 (v(t(1 - x)) + 1 - v(t))_+ dx, \quad v(0) = 0. \quad (6)$$

For the sake of reference, we recall the idea of derivation. Suppose the first mark is x , observed shortly after the start of the process at time $s \in [0, h]$. If x is selected, the mean length of selected subsequence gained by the optimal continuation is $1 + v((t - s)(1 - x))$. If x is rejected, the optimal continuation yields $v(t - s)$. The dynamic programming principle prescribes to select x if and only if $1 + v((t - s)(1 - x)) \geq v(t - s)$, so the better action gives $\max\{1 + v((t - s)(1 - x)), v(t - s)\}$. Integrating out variable x we obtain a recursion

$$v(t) = (1 - h) v(t) + h \int_0^1 \max\{v(t(1 - x)) + 1, v(t)\} dx + o(h),$$

which turns into (6) as $h \rightarrow 0$.

The optimal acceptance window is $\psi^*(t, s, y) = (1 - y) \varphi^*((t - s)(1 - y))$, where $\varphi^*(t) = 1$ if $v(t) \leq 1$ (when $v(t)$ is given by (3)), and otherwise $\varphi^*(t)$ is defined implicitly as the unique solution to

$$v(t(1 - x)) + 1 - v(t) = 0.$$

See [8] for analytic properties and estimates of $v(t)$ and $\varphi^*(t)$ (notably, $v''(t) < 0$). Our focus is on the asymptotic expansion for large t .

With the change of variables $u(z) := v(z^2)$ the optimality equation (6) becomes equation of convolution type

$$u'(z) = 4 \int_0^z (u(z-y) + 1 - u(z))_+(1 - y/z) \, dy, \quad u(0) = 0. \quad (7)$$

We set $\theta^*(z) = z$ if $u(z) \leq 1$, and otherwise define $\theta^*(z)$ to be the unique solution to

$$u(z - y) + 1 - u(z) = 0. \quad (8)$$

Note that $\theta^*(z)$ is always a zero of the integrand. The functions φ^* and θ^* are uniquely related via

$$z - \theta^*(z) = z\sqrt{1 - \varphi^*(z^2)}. \quad (9)$$

By monotonicity we can re-write (7) as

$$u'(z) = 4 \int_0^{\theta^*(z)} (u(z-y) + 1 - u(z))(1 - y/z) \, dy, \quad u(0) = 0. \quad (10)$$

Equation (10) is a special case of the more general equation

$$w'(z) = 4 \int_0^{\theta(z)} (w(z-y) + r(z) - w(z))(1 - y/z) \, dy, \quad w(0) = b, \quad (11)$$

where $r(z)$ and $\theta(z)$ are given functions on $[0, \infty)$, $0 < \theta(z) \leq z$, and b is a constant. Apart from more general inhomogeneous term and initial condition, a major difference between (10) is that the integrand need not be sign-definite, nor should $\theta(z)$ be a zero of the integrand.

Equation (11) has a plausible interpretation in terms of the selection problem. For φ related to θ as in (9), consider the self-similar selection strategy driven by φ . Suppose selecting the generic observation (s, x) yields a reward of the form $r(\sqrt{(t-s)(1-x)})$, and let b be a terminal reward. Define $w(t)$ to be the expected reward accumulated over $[0, t]$. Then w satisfies equation (11). The proof follows by a small- h decomposition, in line with the argument for (6), and a change of variables.

4. A comparison method

Let $\mathcal{I}g$ be the integral operator acting on functions $g \in C^1[0, \infty)$ as

$$\mathcal{I}g(z) = 4 \int_0^z (g(z-y) + 1 - g(z))_+(1 - y/z) \, dy. \quad (12)$$

In this notation equation (7) becomes $u' = \mathcal{I}u$.

The following lemma resembles a familiar comparison method of estimating solutions to differential equations (see [7], section 9.1).

Lemma 1. *If $g'(z) > \mathcal{I}g(z)$ for all sufficiently large z then $\limsup_{z \rightarrow \infty} (u(z) - g(z)) < \infty$. Likewise, if $g'(z) < \mathcal{I}g(z)$ for all sufficiently large z then $\liminf_{z \rightarrow \infty} (u(z) - g(z)) > -\infty$.*

Proof. Observe that $\mathcal{I}g = \mathcal{I}(g + c)$ for constant c . There exists z_0 such that $g'(z) > \mathcal{I}g(z)$ for $z > z_0$. Assume to the contrary that $\limsup_{z \rightarrow \infty} (u(z) - g(z)) = \infty$. Choose c large enough to achieve that $z_1 := \min\{z : u(z) = g(z) + c\}$ satisfies $z_1 > z_0$. This is possible, since by the assumption z_1 is well defined for every $c > u(0) - g(0)$ and $z_1 \rightarrow \infty$ as $c \rightarrow \infty$. Then for $y < z_1$,

$$u(z_1 - y) + 1 - u(z_1) \leq (g(z_1 - y) + c) + 1 - (g(z_1) + c), \quad (13)$$

whence

$$(u(z_1 - y) + 1 - u(z_1))_+ \leq ((g(z_1 - y) + c) + 1 - (g(z_1) + c))_+,$$

and my monotonicity of the integral $u'(z_1) = \mathcal{I}u(z_1) \leq \mathcal{I}(g + c)(z_1) = \mathcal{I}g(z_1) < g'(z_1)$. But this is a contradiction since $u'(z_1) \geq (g + c)'(z_1) = g'(z_1)$ by definition of z_1 as the location where u first reaches $g + c$. The second part of the lemma is argued similarly. \square

Note that in (12) it is sufficient to integrate over the range $\{y \in [0, z] : g(z - y) + 1 - g(z) > 0\}$ which depends on g . In contrast to that, the (nonhomogeneous, linear) integral operator

$$\mathcal{J}g(z) := 4 \int_0^{\theta(z)} (g(z - y) + r(z) - g(z))(1 - y/z) \, dy,$$

appearing in equation (11) involves integration with fixed limits.

Lemma 2. *Let w be a solution to (11) for some continuous functions θ and r . If $g'(z) > \mathcal{J}g(z)$ for all sufficiently large z then $\limsup_{z \rightarrow \infty} (w(z) - g(z)) < \infty$. Likewise, if $g'(z) < \mathcal{J}g(z)$ for all sufficiently large z then $\liminf_{z \rightarrow \infty} (w(z) - g(z)) > -\infty$.*

Proof. Modifying the argument in the previous lemma replace (13) with

$$w(z_1 - y) + r(z_1) - u(z_1) \leq (g(z_1 - y) + c) + r(z_1) - (g(z_1) + c),$$

and skip taking the positive part thereafter. \square

5. Asymptotics I

We will now compare the solution to (7) with various test functions. Let $u_1(z) := \alpha_1 z$. We have $u_1'(z) = \alpha_1$ and for $\theta_1(z) := 1/\alpha_1$,

$$\mathcal{I}u_1(z) = 4 \int_0^{\theta_1(z)} (u_1(z - y) + 1 - u_1(z))(1 - y/z) \, dy \rightarrow \frac{2}{\alpha_1}, \quad z \rightarrow \infty.$$

The match $\alpha_1 = 2/\alpha_1$ occurs at $\alpha_1^* := \sqrt{2}$, thus by Lemma 1, $\limsup_{z \rightarrow \infty} (u(z) - u_1(z)) < \infty$ for $\alpha_1 > \sqrt{2}$ and therefore $\limsup_{z \rightarrow \infty} u(z)/z \leq \sqrt{2}$. Likewise, the second part of the lemma yields $\liminf_{z \rightarrow \infty} u(z)/z \geq \sqrt{2}$. These bounds imply $u(z) \sim \sqrt{2}z$.

We try next functions $u_2(z) := \sqrt{2}z + \alpha_2 \log(z+1)$ (we take $\log(z+1)$ and not $\log z$ to avoid the annoying singularity at 0). Solving $u_2(z-y) + 1 - u_2(z) = 0$, for large z we get expansion

$$\theta_2(z) \sim \frac{1}{\sqrt{2}} - \frac{\alpha_2}{2(z+1)}. \quad (14)$$

We may proceed with only the first term in (14) since the second makes a negligible $O(z^{-2})$ contribution to $\mathcal{I}u_2(z)$ which expands as

$$\mathcal{I}u_2(z) \sim \sqrt{2} - \left(\frac{1}{3} + \alpha_2\right) \frac{1}{z+1}.$$

With $u'_2(z) = \sqrt{2} + \alpha_2/(z+1)$ the match occurs when

$$\alpha_2 = -\left(\frac{1}{3} + \alpha_2\right),$$

that is for $\alpha_2^* := -1/6$. It follows from Lemma 1 that $(u(z) - \sqrt{2}z)/\log(z+1) \rightarrow \alpha_2^*$, that is

$$u(z) \sim \sqrt{2}z - \frac{1}{6} \log z.$$

To further refine the approximation we try

$$u_3(z) := \sqrt{2}z - \frac{1}{6} \log(z+1) + \frac{\alpha_3}{z+1}. \quad (15)$$

This time we need to calculate with higher precision, hence take two terms

$$\theta_3(z) \sim \frac{1}{\sqrt{2}} + \frac{1}{12(z+1)}. \quad (16)$$

Expanding the integrand and integrating:

$$\mathcal{I}u_3(z) \sim \sqrt{2} - \frac{1}{6(z+1)} + \left(\alpha_3 - \frac{1}{36\sqrt{2}} - \frac{1}{3}\right) \frac{1}{(z+1)^2}.$$

To match with

$$u'_3(z) = \sqrt{2} - \frac{1}{6(z+1)} - \frac{\alpha_3}{(z+1)^2}$$

we must choose $\alpha_3^* := 1/6 + \sqrt{2}/144$. Taking α_3 bigger or smaller than α_3^* , allows us to sandwich u . However, our comparison method based on Lemma 1 only yields

$$u(z) = \sqrt{2}z - \frac{1}{6} \log z + O(1), \quad z \rightarrow \infty, \quad (17)$$

since the third term in (15) is already bounded. A different approach will be applied to show convergence of the $O(1)$ remainder.

6. Piecewise deterministic Markov process

Suppose a self-similar strategy is employed. If y is the running maximum at time s , the distribution of the number of selections to follow only depends on the process past through $(1-y)(t-s)$. This suggests to merge the running maximum and the observation time in one parameter and to study its evolution. Adopting $z = \sqrt{(1-y)(t-s)}$ as a state variable and introducing an intrinsic time variable will lead us to a nearly homogeneous Markov process which we denote Z .

Let $\theta : (0, \infty) \rightarrow (0, \infty)$ be a function satisfying $0 < \theta(z) \leq z$, and let

$$\lambda(z) := \theta(z) - \frac{\theta^2(z)}{2z}.$$

The following rules define a piecewise deterministic Markov process Z on $[0, \infty)$ with continuous drift component and random instantaneous jumps:

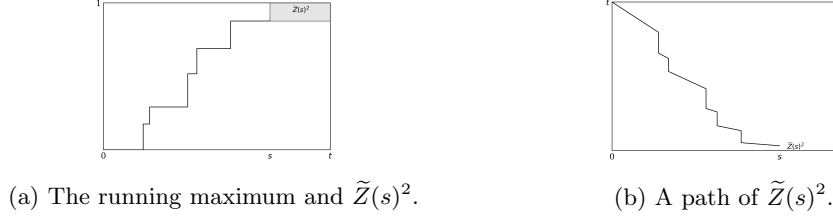
- (i) the process decreases continuously with unit speed,
- (ii) the jumps are negative and occur at rate $4\lambda(z)$, for $z > 0$,
- (iii) if a jump from state z occurs, the jump size has density $(1-y/z)/\lambda(z)$ with support $[0, \theta(z)]$,
- (iv) the process terminates upon reaching 0.

We denote $Z|z_0$ this process starting in position z_0 . The range of $Z|z_0$ can be constructed from the set of arrivals of an inhomogeneous marked Poisson process Π with intensity (ii) and marks distributed as in (iii). The following occupancy procedure is similar to many familiar parking, packing and scheduling models in applied probability. With each occurrence z of Π marked y relate interval $(z-y, z]$. Now, moving right-to-left from z_0 create a non-overlapping configuration by leaving the rightmost $(z_1 - y_1, z_1]$ in its position and removing all other intervals that overlap this one, then proceed this way to the left of $z_1 - y_1$ until reaching 0. The process $Z|z_0$ crosses each $(z_j - y_j, z_j]$ by jump, and drifts through the rest of $[0, z_0]$. A location $z \in (0, z_0)$ is called a *jump point* if $z \in \{z_j, j \geq 1\}$, a *gap point* if $z \in \cup_j (z_j - y_j, z_j]$ and a *drift point* otherwise. For the corresponding path of $Z|z_0$ there is a unique way to introduce the time variable in agreement with rule (i). Specifically, the time when $Z|z_0$ reaches z is equal to the Lebesgue measure of the set of drift points within $[z, z_0]$. The path is naturally decomposed in *cycles*, each comprised of a drift interval and a jump interval in the right-to-left succession. The rightmost cycle is $(z_1 - y_1, z_1] \cup (z_1, z_0]$, and the leftmost cycle has only a drift interval.

To connect to the increasing subsequence problem fix horizon t and let Y be the running maximum process under some self-similar strategy (5). Let

$$\tilde{Z}(s) := \sqrt{(1-Y(s))(t-s)}, \quad s \in [0, t],$$

which is a drift-jump process decreasing from $t^{1/2}$ to 0, with negative jumps $\Delta\tilde{Z}(s) = \tilde{Z}(s) - \tilde{Z}(s-)$ at times of selection. Figure 1 illustrates the correspondence.

Figure 1: Transformation of Y to \tilde{Z} .

We wish to replace the observation time s by an intrinsic time parameter associated with drift. To that end, first note that the decay of \tilde{Z} due to the drift is a strictly increasing continuous process

$$\sigma(s) := t^{1/2} - \tilde{Z}(s) + \sum_{s' \leq s} \Delta \tilde{Z}(s').$$

For σ^\leftarrow the inverse function to σ , define the time-changed process

$$Z(q) := \tilde{Z}(\sigma^\leftarrow(q)), \quad q \leq \sigma(t). \quad (18)$$

Identifying the drift rate and jump distribution it is seen that (18) is the process $Z|\sqrt{t}$, with θ found by matching the jump rates as

$$4\lambda(z) = 2z\varphi(z^2).$$

In particular, Y over horizon $t = z^2$ has the same number of jumps as $Z|z$. This reduces the optimal selection problem with horizon t to choosing a control function θ with the objective to maximise the expected number of jumps of $Z|\sqrt{t}$.

Denote $N_\theta(z)$ the number of jumps of the process $Z|z$ steered by given function θ ($0 < \theta(z) \leq z$), and let $u_\theta(z) := \mathbb{E}N_\theta(z)$. With probability $4\lambda(z)dz$ the process moves from a small vicinity of z to $z - y$, with y sampled from the density in (iii), in which case the expected number of jumps is equal to $u_\theta(z - y) + 1$. Otherwise, the process drifts through to $z - dz$. This decomposition readily yields equation

$$u'_\theta(z) = 4 \int_0^{\theta(z)} (u_\theta(z - y) + 1 - u_\theta(z))(1 - y/z) dy, \quad u_\theta(0) = 0, \quad (19)$$

which is a special case of (11) derived earlier in the context of the running maximum Y . In purely analytic terms, for any fixed z , maximising $u_\theta(z)$ over admissible θ is the problem of calculus of variations. The solution is $\theta = \theta^*$, defined implicitly by equations (7) and (8).

We shall assume throughout that θ is bounded and differentiable. That the optimal θ^* is bounded can be seen at this stage of our analysis from (8) and (17).

The asymptotic comparison method, now based on Lemma 2 works for (19) smoothly. In particular, for

$$\theta_0(z) := \sqrt{1/2} \wedge z,$$

we obtain the same expansion as (17). Complementing this technique, we will adopt some ideas from the potential theory for Markov processes.

The decreasing sequence of jump points of $Z|_{z_0}$ is a an embedded Markov chain with terminal state 0. Let $U_\theta(z_0, \cdot)$ be the occupation measure on $[0, z_0]$ counting the expected number of jump points, in particular $U_\theta(z, [0, z]) = u_\theta(z)$. Denote $p(z_0, z)$, for $0 \leq z \leq z_0$ the probability that z is a drift point, in particular $p(z_0, z_0) = p(z_0, 0) = 1$. There is a jump point within dz only if z does not belong to a gap, hence the occupation measure has a density which factorises as

$$U_\theta(z_0, dz) = 4\lambda(z) p(z_0, z) dz, \quad 0 \leq z \leq z_0.$$

Lemma 3. *There exists a pointwise limit $p(z) := \lim_{z_0 \rightarrow \infty} p(z_0, z)$, which satisfies*

$$|p(z_0, z) - p(z)| < ae^{-\alpha(z_0 - z)}, \quad 0 < z < z_0,$$

with some positive constants a and α .

Proof. The proof is by coupling. Choose constant $\bar{\theta}$ big enough to have $\sup \theta(z) < \bar{\theta}$. Fix $z < z_0 < z_1$ with $z > 2\bar{\theta}$ (the latter assumption does not affect the result). Consider two independent processes Z_0 and Z_1 with $Z_0 \stackrel{d}{=} Z|_{z_0}$, $Z_1 \stackrel{d}{=} Z|_{z_1}$. Define Z' by running the process Z_1 until it hits a drift point ξ of Z_0 , then from this point on switch over to running Z_0 . Such a point ξ exists since both processes have a gap adjacent to 0. By the strong Markov property, Z' has the same distribution as Z_1 . If the coupling occurs at some $\xi \in [z, z_0]$, the point z is of the same type (drift or jump) for both Z' and Z_0 .

The coupling does not occur within $[z, z_0]$ only if Z_0 and Z_1 have no common drift points within these bounds. Given that $y > z$ is a drift point, the probability that the drift interval covering y extends to the left over $y - \bar{\theta}$ is at least π , for some constant $\pi > 0$. This follows since the length of drift interval dominates stochastically an exponential random variable with rate $\sup 4\lambda(z) < \infty$. In particular, the rightmost drift interval, adjacent to z_0 , is shorter than $\bar{\theta}$ with probability at most $1 - \pi$, in which case the rightmost cycle is shorter than $2\bar{\theta}$. Given ξ is not in the first cycle, the probability that ξ is not in the second is again at most $1 - \pi$, in which case also the second cycle is shorter than $2\bar{\theta}$. Continuing so forth we see that $\xi \notin [z, z_0]$ with probability at most $(1 - \pi)^k$ for $k = \lfloor (z_0 - z)/(2\bar{\theta}) \rfloor$. This readily implies an exponential bound $|p(z_0, z) - p(z_1, z)| < ae^{-\alpha(z_0 - z)}$, uniformly in $z_1 > z_0$. Sending $z_0 \rightarrow \infty$ we see that $p(z_0, z)$ is a Cauchy sequence, whence the claim. \square

In the terminology of random sets, $p(z_0, \cdot)$ is the coverage function (see [20] p. 23) for the range of $Z|_{z_0}$. As $z_0 \rightarrow \infty$ the range converges weakly to a random set $\mathcal{Z} \subset [0, \infty)$, comprised of infinitely many intervals separated by gaps. Indeed, let $A(z_0, z) \leq z$ be the

maximal point of the range of $Z|z_0$ within $[0, z]$, for $z \leq z_0$. The coupling argument in the lemma also shows that $A(z_0, z)$ has a weak limit, $A(z)$, which is sufficient to justify convergence of the range intersected with $[0, z]$, due to the Markov property. By Sheffé's lemma $U_\theta(z_0, \cdot)$ converges weakly to some U_θ , which is the occupation measure for the point process of left endpoints of intervals making up \mathcal{Z} .

7. Reward processes

Suppose each jump point of $Z|z$ is weighted by some location-dependent reward r . Let $w_{\theta,r}(z)$ be the total expected reward accumulated by $Z|z$ controlled by θ . This is, of course, the same function as the one we discussed when deriving (11). But now we also have an integral representation of $w_{\theta,r}(z)$ as the average over the occupation measure,

$$w_{\theta,r}(z) = \int_0^z r(y) U_\theta(z, dy) = 4 \int_0^z r(y) \lambda(y) p_\theta(z, y) dy. \quad (20)$$

Lemma 4. *For r integrable function, the solution to (11) has a finite limit*

$$\rho_{\theta,r} := \lim_{z \rightarrow \infty} w_{\theta,r}(z) = 4 \int_0^\infty r(y) \lambda(y) p_\theta(y) dy. \quad (21)$$

If $|r(z)| = O(z^{-\beta})$ as $z \rightarrow \infty$ for some $\beta > 1$ then $|w_{\theta,r}(z) - \rho_{\theta,r}| = O(z^{-\beta+1})$.

Proof. Since $p(z_0, z) \lambda(z) < \bar{\theta}$ the existence of limit follows from (20), (21) and Lemma 3 by the dominated convergence. The convergence rate is estimated by splitting the difference as

$$\rho_{\theta,r} - w_{\theta,r}(z) = 4 \int_0^{z/2} r(y) \lambda(y) (p_\theta(z, y) - p_\theta(y)) dy + 4 \int_{z/2}^\infty r(y) \lambda(y) p_\theta(y) dy,$$

where the second integral is of the order $O(z^{-\beta+1})$ while the first is of the lesser order $O(e^{-\alpha z/2})$ by Lemma 3. \square

8. Asymptotics II

We are ready to derive finer asymptotics. Let

$$r(z) := \frac{4}{\lambda(z) z^2} \int_0^{\theta^*(z)} (u(z-y) + 1 - u(z)) y dy.$$

Differentiating (7) and keeping an account of (8), we obtain

$$u''(z) = 4 \int_0^{\theta^*(z)} (u'(z-y) + r(z) - u'(z))(1 - y/z) dy, \quad u'(0) = 0. \quad (22)$$

Since $\theta^*(z) = z$ for small z this has a simple pole at 0, but the singularity is compensated in (20), so Lemma 4 and (17) ensure that

$$u'(z) = \sqrt{2} + O(z^{-1}). \quad (23)$$

With (23) at hand, expanding the solution to (8) we get

$$\theta^*(z) = \sqrt{1/2} + O(z^{-1}).$$

Replacing θ^* by $\sqrt{1/2}$ in (7) incurs remainder of smaller order $O(z^{-2})$ because $\theta^*(z)$ is the stationary point of the integral viewed as a function of the upper bound. Recalling that u_2 (with $\alpha_2^* = -1/6$) satisfies $u_2'(z) = \mathcal{I}u(z) + O(z^{-2})$, for the difference $w = u - u_2$ we obtain equation (11) with $r(z) = O(z^{-2})$, hence $u(z) - u_2(z)$ by Lemma 4 approaches a finite limit at rate $O(z^{-1})$ as $z \rightarrow \infty$. This proves an expansion

$$u(z) = \sqrt{2}z - \frac{1}{6} \log z + c^* + O(z^{-1}), \quad z \rightarrow \infty \quad (24)$$

with some constant c^* .

Our methods are not geared to identify c^* , because the initial value $u(0) = 0$ was nowhere used, but changing it to $u(0) = b$ (which is resorting to a selection problem with terminal reward b) will result in adding b to c^* . Nevertheless, with some more effort it is possible to go beyond $O(1)$. Let us first estimate the local variation of u' .

Lemma 5. *For fixed $\bar{d} > 0$, as $z \rightarrow \infty$*

$$\sup_{0 \leq d \leq \bar{d}} |u'(z+d) - u'(z)| = O(z^{-2}).$$

Proof. Using the integral representation (20) of u' with $r(z) = O(z^{-2})$, write

$$u'(z+d) - u'(z) = \int_z^{z+d} r(y) p(z+d, y) \lambda(y) dy + \int_0^z |p(z+d, y) - p(z, y)| \lambda(y) r(y) dy.$$

The first integral is obviously $O(z^{-2})$ uniformly in $d \leq \bar{d}$. By Lemma 3 the second is estimated as

$$c \int_0^z e^{-\alpha(z-y)} (y^2 + 1)^{-1} dy = O(z^{-2})$$

using Laplace's method. □

The lemma applied to the right-hand side of (22) gives $u''(z) = O(z^{-2})$. In (10) we replace θ^* by $\sqrt{1/2}$, expand $u(z-y) - u(z) = -yu'(z) + O(z^{-2})$ and integrate to obtain with some algebra

$$u'(z) = \sqrt{2} - \frac{1}{6z} + O(z^{-2}).$$

Expanding similarly in (8) we get a finer formula for the optimal control function

$$\theta^*(z) = \frac{1}{\sqrt{2}} + \frac{1}{12z} + O(z^{-2}), \quad z \rightarrow \infty, \quad (25)$$

in accord with (16). Since $u'_3(z) = \mathcal{I}u_3(z) + O(z^{-3})$ the difference $w = u - u_3$ satisfies (11) with $r(z) = O(z^{-3})$, hence invoking Lemma 4 we obtain $u(z) - u_3(z) = \hat{c} + O(z^{-2})$ for some constant \hat{c} . This must agree with (24), therefore $\hat{c} = c^*$. Thus we have shown

Theorem 6. *For the optimal process, the control function θ^* satisfies (25), and the expected number of jumps has expansion*

$$u(z) = \sqrt{2}z - \frac{1}{6} \log z + c^* + \frac{\sqrt{2}}{144z} + O(z^{-2}), \quad z \rightarrow \infty. \quad (26)$$

To appreciate the effect of the second term in (25) it is helpful to consider control functions of the kind

$$\theta(z) \sim \frac{1}{\sqrt{2}} + \frac{\gamma}{z}, \quad z \rightarrow \infty. \quad (27)$$

The parameter appears in the asymptotics of solutions to (19) as

$$u_\theta(z) \sim \sqrt{2}z - \frac{1}{6} \log z + c + \left(\frac{\sqrt{2}}{72} - \frac{\sqrt{2}\gamma}{6} + \sqrt{2}\gamma^2 \right) \frac{1}{z} \quad (28)$$

(whichever $u_\theta(0)$ that only affects the constant).

Constant c in (28) does not exceed c^* in (26), but the relation between the z^{-1} -terms can be the opposite. For instance, for $\theta_0(z) = \sqrt{1/2} \wedge z$ we have

$$u_{\theta_0}(z) = \sqrt{2}z - \frac{1}{6} \log z + c_0 + \frac{\sqrt{2}}{72z} + O(z^{-2}).$$

9. The variance

For $N_\theta(z)$, the number of jumps of $Z|z$ driven by θ , let $w(z) = \mathbb{E}(N_\theta(z))^2$ be the second moment. This function satisfies

$$w'(z) = 4 \int_0^{\theta(z)} [w(z-y) + (1 + 2u_\theta(z-y)) - w(z)](1 - y/z) \, dy, \quad w(0) = 0.$$

Integrating the inhomogeneous term this can be reduced to the form (11), with $r(z)$ of the order of z . Applying Lemma 2 we compare w with various test functions.

We shall consider first the case of optimal $\theta = \theta^*$. It is an easy exercise to see that $w(z) \sim 2z^2$, hence the leading term in the integrand is $-4zy + 2\sqrt{2}z$, which vanishes at $y = \sqrt{1/2}$. For this reason the $O(z^{-2})$ remainder in (25) will contribute to the solution

only $O(1)$, and not $O(\log z)$ as one might expect. Using this fact and (25) it is possible to match the sides of the equation by selecting coefficients of the test function

$$\hat{w}(z) = 2z^2 + a_1 z \log z + a_2 z + a_3 (\log z)^2 + a_4 \log z,$$

achieving that the difference $w(z) - \hat{w}(z)$ satisfies an equation of the type (11) with $r(z) = O(z^{-2} \log z)$. Then applying Lemma 4, $w(z) - \hat{w}(z) \sim c_1 + z^{-1} \log z$. With some help of **Mathematica** we arrived at

$$w(z) \sim 2z^2 - \frac{\sqrt{2}}{3} z \log z + \left(\frac{\sqrt{2}}{3} + 2\sqrt{2}c^* \right) z + \frac{1}{36} (\log z)^2 + \left(\frac{1}{36} - \frac{c^*}{3} \right) \log z + c_1.$$

From this and (24) for $\text{Var}(N_{\theta^*}(z)) = w(z) - u^2(z)$ we obtain

$$\text{Var}(N_{\theta^*}(z)) = \frac{\sqrt{2}z}{3} + \frac{1}{36} \log z + c_2 + O(z^{-1} \log z), \quad z \rightarrow \infty.$$

with $c_2 := c_1 - (c^*)^2 - 1/36$. In fact, the value of c^* in (24) impacts c_1 but not c_2 , because the latter is invariant under shifting $u(0)$.

For the general control functions, the variance is very sensitive to the behaviour of θ . The convergence $\theta(z) \rightarrow \sqrt{1/2}$ alone does not even ensure that $O(z)$ is the right *order* for $\text{Var}(N_{\theta^*}(z))$. If (27) holds we have the asymptotics

$$\text{Var}(N_{\theta}(z)) \sim \frac{\sqrt{2}z}{3} + \left(\frac{1-8\gamma}{12} \right) \log z, \quad z \rightarrow \infty.$$

10. CLT for the number of jumps

If the control function $\theta(z)$ approaches a constant for large z , the process Z afar from 0 is almost homogeneous. This suggests approximating the path of Z by a decreasing renewal process with two types of decrements corresponding to drift intervals and gaps.

In this section we denote $N(z)$ the number of jumps of $Z|z$ with some control function satisfying

$$\theta(z) = \frac{1}{\sqrt{2}} + O(z^{-1}), \quad \text{hence } \lambda(z) = \theta(z) - \frac{\theta^2(z)}{2z} = \frac{1}{\sqrt{2}} + O(z^{-1}), \quad z \rightarrow \infty. \quad (29)$$

Denote J_z the size of the generic gap having the right endpoint z , with density

$$\mathbb{P}(J_z \in dy) = \frac{1 - y/z}{\lambda(z)}, \quad 0 \leq y \leq \theta(z),$$

and let D_z be the size of the generic drift interval with survival function

$$\mathbb{P}(D_z \geq y) = \exp \left(- \int_{z-y}^z 4\lambda(s) \, ds \right), \quad 0 \leq y \leq z. \quad (30)$$

The size of the generic cycle with the right endpoint z can be written as

$$D_z + J_{z-D_z},$$

where D_z and the family of variables J_\cdot are independent, and we set $J_0 = 0$.

For large z , the expected values of J_z and D_z are about equal, suggesting that about a half of $[0, z]$ is covered by drift and another half is skipped by jumps. This resembles the behaviour of the stationary (and, as seen from [9], also of the optimal) selection process in the planar Poisson setting, where the balance is kept on two scales.

It is useful to see how the mean sizes of gaps and drift intervals depend on $\theta = \theta(z)$:

$$\mathbb{E}J_z = \frac{\theta}{2} - \frac{\theta^2}{12z} + O(z^{-2}), \quad \mathbb{E}D_z = \frac{1}{4\theta} + \frac{1}{8z} + O(z^{-2}).$$

For θ as in (29), the mean size of a cycle is

$$\mathbb{E}(J_z + D_z) \sim \frac{1}{\sqrt{2}} + \frac{1}{12z} + O(z^{-2}),$$

regardless of the $O(z^{-1})$ term in (29). This expansion explains why the second term in (26) is $O(\log z)$ (but falls short of explaining the coefficient $-1/6$), and why the suboptimal strategy in Theorem 6 is $O(1)$ from the optimum.

From the convergence of parameters (29) it is clear that as $z \rightarrow \infty$

$$D_z \xrightarrow{d} \frac{E}{2\sqrt{2}}, \quad J_z \xrightarrow{d} \frac{U}{\sqrt{2}},$$

and, observing the joint convergence of (D_z, J_{z-D_z}) , also that

$$D_z + J_{z-D_z} \xrightarrow{d} \frac{E}{2\sqrt{2}} + \frac{U}{\sqrt{2}}, \tag{31}$$

where $U \stackrel{d}{=} \text{Uniform}[0, 1]$ and $E \stackrel{d}{=} \text{Exponential}(1)$ are independent.

The weak convergence (31) of cycle sizes suggests that the behaviour of $N(z)$ for large z can be deduced from that of a renewal process with the generic step

$$H := \frac{E}{2\sqrt{2}} + \frac{U}{\sqrt{2}}$$

which has moments

$$\mu := \mathbb{E}H = \frac{1}{\sqrt{2}}, \quad \sigma^2 := \text{Var}(H) = \frac{1}{6}, \quad \frac{\sigma^2}{\mu^3} = \frac{\sqrt{2}}{3}.$$

Specifically, for the renewal process $R(z) := \max\{n : H_1 + \dots + H_n \leq z\}$, with H_j 's being i.i.d. replicas of H , we have the familiar CLT

$$\frac{R(z) - z\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \xrightarrow{d} \mathcal{N}(0, 1),$$

and one can expect that the same limit holds for $N(z)$. This line should be pursued with care, because local discrepancies may accumulate on the large scale and bias centring or even the type of the limit distribution.

Our search of the literature on nonlinear renewal theory to cover the situation of interest showed that the best relevant work is due to Cutsem and Ycart [13]. Their setting of lattice processes is easy to modify, but the argument in [13] has a gap and, in fact, the main result fails without additional assumptions (see a remark below). In the approach taken here, we amend some details of their method of stochastic comparison. To that end, with initial state $z \rightarrow \infty$, we focus on the cycles that lie within some range $[\underline{z}, z]$, where the truncation parameter \underline{z} is properly chosen to warrant approximation of the whole process.

The asymptotics (29) implies that there exists a constant $c > 0$ such that for all sufficiently large \underline{z} the parameters can be bounded as

$$\begin{aligned} \frac{1 - c/\underline{z}}{\sqrt{2}} &< \lambda(z) < \frac{1 + c/\underline{z}}{\sqrt{2}}, \\ \frac{1}{\sqrt{2}(1 + c/\underline{z})} &< \theta(z) < \frac{1}{\sqrt{2}(1 - c/\underline{z})} \end{aligned}$$

uniformly in $z > \underline{z}$. Replacing the variable rate in (30) by constant yields the bounds

$$\left((1 + c/\underline{z})^{-1} \frac{E}{2\sqrt{2}} \right) \wedge (z - \underline{z}) <_{\text{st}} D_z \wedge (z - \underline{z}) <_{\text{st}} (1 - c/\underline{z})^{-1} \frac{E}{2\sqrt{2}},$$

where and henceforth $<_{\text{st}}$ denotes the stochastic order. Observing that the survival function of J_z is convex, we may bound the jump as

$$\lambda(z)U <_{\text{st}} J_z <_{\text{st}} \theta(z)U,$$

whence from (32)

$$(1 + c/\underline{z})^{-1} \frac{U}{\sqrt{2}} <_{\text{st}} J_z <_{\text{st}} (1 - c/\underline{z})^{-1} \frac{U}{\sqrt{2}}, \quad z \geq \underline{z}.$$

From these estimates follow stochastic bounds on the cycle size

$$((1 + c/\underline{z})^{-1}H) \wedge (z - \underline{z}) <_{\text{st}} (D_z + J_{z-D_z}) \wedge (z - \underline{z}) <_{\text{st}} (1 - c/\underline{z})^{-1}H, \quad z \geq \underline{z}. \quad (32)$$

Setting the bounds (32) in terms of multiples of the same random variable H is convenient in combination with the obvious scaling property: for $d > 0$, $R(d\cdot)$ is the renewal process with the generic step dH . Let $N(z, \underline{z})$ be the number of cycles of $Z|z$, which fit completely within $[\underline{z}, z]$. As in [13], from (32) we conclude that

$$R((z - \underline{z})(1 - c/\underline{z})) <_{\text{st}} N(z, \underline{z}) <_{\text{st}} R((z - \underline{z})(1 + c/\underline{z})), \quad z \geq \underline{z}. \quad (33)$$

Letting $z \rightarrow \infty$ then $\underline{z} \rightarrow \infty$, and appealing to $R(z)/z \rightarrow \mu^{-1}$ a.s., (33) implies a weak law of large numbers for $N(z)$,

$$\frac{N(z)}{z} \xrightarrow{d} \frac{1}{\mu}, \quad z \rightarrow \infty. \quad (34)$$

We aim next to show the CLT for $N(z)$, that is

$$\frac{N(z) - z\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \xrightarrow{d} \mathcal{N}(0, 1), \quad z \rightarrow \infty. \quad (35)$$

To that end, we choose $\underline{z} = \omega\sqrt{z}$, where $\omega > 0$ is a large parameter. Start with splitting

$$N(z) - z\mu^{-1} = (N(z, \underline{z}) - (z - \underline{z})\mu^{-1}) + (N(z) - N(z, \underline{z}) - \underline{z}\mu^{-1}),$$

where $N(z) - N(z, \underline{z})$ counts the cycles that start in $[0, \underline{z}]$; this component is annihilated by the scaling, since by (34)

$$\frac{N(z) - N(z, \underline{z}) - \underline{z}\mu^{-1}}{\sqrt{\underline{z}}} \xrightarrow{d} 0,$$

and the same is true with $\sqrt{\underline{z}}$ replaced by bigger \sqrt{z} . For the leading contribution due to $N(z, \underline{z})$ we obtain using dominance (33) and the CLT for $R(z)$

$$\begin{aligned} \mathbb{P}\left(\frac{N(z, \underline{z}) - (z - \underline{z})\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \leq x\right) &\geq \mathbb{P}\left(\frac{R((z - \underline{z})(1 + c/\underline{z})) - (z - \underline{z})\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \leq x\right) = \\ &\mathbb{P}\left(\frac{R((z - \underline{z})(1 + c/\underline{z})) - (z - \underline{z})(1 + c/\underline{z})\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} + \frac{(z - \underline{z})c}{\omega z\mu^{-1/2}\sigma} \leq x\right) \\ &\rightarrow 1 - \Phi\left(x - \frac{c}{\omega\sigma\mu^{-1}}\right), \end{aligned}$$

as $z \rightarrow \infty$. Letting $\omega \rightarrow \infty$

$$\limsup_{z \rightarrow \infty} \mathbb{P}\left(\frac{N(z) - z\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \leq x\right) = \limsup_{z \rightarrow \infty} \mathbb{P}\left(\frac{N(z, z_0) - z\mu^{-1}}{\sigma\mu^{-3/2}\sqrt{z}} \leq x\right) \geq 1 - \Phi(x).$$

The opposite inequality is derived similarly. Hence (35) is proved.

Remark. The renewal-type approximation for decreasing Markov chains on \mathbb{N} , using stochastic comparison appeared in [13]. However, their Theorem 4.1 on the normal limit for the absorption time fails without additional assumptions on the quality of convergence of the step distribution. For instance, if the decrement in position $z > 8$ assumes values 1 and 2 with probabilities $1/2 \pm 1/\log z$, the mean absorption time is asymptotic to $2z/3$, with the remainder being strictly of the order $z/\log z$, therefore not annihilated by the \sqrt{z} scaling. The error in [13] appears on the bottom of page 996, where the truncation parameter (m , a counterpart of our \underline{z}) is assumed independent of the initial state. Recently Alsmeyer and Marynych [1], also concerned with the lattice setting, suggested conditions on the rate of convergence of decrements in some probability metrics, to ensure the normal approximation of the absorption time.

Remark. It is of interest to look at the properties of the random set \mathcal{Z} which, intuitively, describes an infinite selection process. This limit object can be interpreted in the

spirit of the boundary theory of Markov processes: the state space $[0, \infty)$ has a one-point compactification - the entrance Martin boundary - approached as the initial state of $Z|z$ tends to ∞ . Applying the coupling argument as in Lemma 3 one can show that, at large distance from the origin, Z behaves similarly to a stationary alternating renewal process, with uniformly distributed gaps and exponential drift intervals. The coverage probability and the occupation measure satisfy $p(z) \rightarrow 1/2$ and $U([0, z]) \sim \sqrt{2}z$, $z \rightarrow \infty$. Korshunov [19] studied increasing Markov processes on reals which at distance from the origin behave similarly to renewal processes, but reverting the direction of time, required to adapt this work in our setting, does not seem straightforward.

11. Summary

We summarise our findings in terms of the original problem. As before, let $L_\varphi(t)$ be the length of increasing subsequence selected by a self-similar strategy with the acceptance window of the form (5).

Theorem 7.

- (a) *The optimal strategy has the acceptance window of the form (5) with*

$$\varphi^*(t) = \sqrt{\frac{2}{t}} - \frac{1}{3t} + O(t^{-3/2}), \quad t \rightarrow \infty,$$

and outputs an increasing subsequence with expected length

$$v(t) = \mathbb{E}L_{\varphi^*}(t) = \sqrt{2t} - \frac{1}{12} \log t + c^* + \frac{\sqrt{2}}{144\sqrt{t}} + O(t^{-1}), \quad t \rightarrow \infty,$$

and variance

$$\text{Var}(L_{\varphi^*}(t)) \sim \frac{\sqrt{2t}}{3} + \frac{1}{72} \log t + c_2 + O(t^{-1/2} \log t).$$

- (b) *The strategy with $\varphi_0(t) := \sqrt{2/t} \wedge 1$ outputs an increasing subsequence with the expected length*

$$\mathbb{E}L_{\varphi_0}(t) = \sqrt{2t} - \frac{1}{12} \log t + c_0 + \frac{\sqrt{2}}{72\sqrt{t}} + O(t^{-1}), \quad t \rightarrow \infty,$$

and variance

$$\text{Var}(L_{\varphi_0}(t)) \sim \frac{\sqrt{2t}}{3} + \frac{1}{24} \log t + c_3 + O(t^{-1/2} \log t), \quad t \rightarrow \infty.$$

- (c) *If $\varphi(t) \sim \sqrt{2/t} + O(t^{-1})$ then a central limit theorem holds:*

$$\sqrt{3} \frac{L_\varphi(t) - \sqrt{2t}}{(2t)^{1/4}} \xrightarrow{d} \mathcal{N}(0, 1), \quad t \rightarrow \infty.$$

The instance of part (c) for the optimal strategy was proved in [9]; this can be compared with the distributional limit (4) for the stationary strategy.

Bruss and Delbaen [9] used concavity of v to prove the bounds

$$\frac{v(t)}{3} \leq \text{Var}(L_{\varphi^*}(t)) \leq \frac{v(t)}{3} + \frac{1}{(\beta - \sqrt{2\beta})6\sqrt{2}} \log \frac{t}{\beta} + 2,$$

(for t no too small), where $v(\beta) = 2$. For large t , the logarithmic term in the lower bound has coefficient $-1/36$ (as is seen from (a)) and in the upper bound at least 0.55 (as can be shown by estimating β). These bounds can be compared with the precise coefficient $1/72$ in part (a).

Remark. The version of the problem with a fixed number of observations n is more complex, because the time of observation m and the running maximum y cannot be aggregated in a single state variable [3, 25]. Nevertheless, one can expect that the value function is well approximable by a function of $(n - m + 1)(1 - y)$, hence an analogue of self-similar strategy in Theorem 7 (b), that is the strategy with acceptance condition

$$0 < \frac{x - y}{1 - y} < \sqrt{\frac{2}{(n - m + 1)(1 - y)}} \bigwedge 1, \quad (36)$$

is close to optimality. Arlotto et al. [3] employed (1) and a de-Poissonisation argument to show that, indeed, the strategy given by (36) is within $O(\log n)$ from the optimum for n large. By extending the methods of the present paper, the latter result has been strengthened recently in [26] : the expected length of subsequence selected with acceptance window (36) is $\sqrt{2n} - (\log n)/12 + O(1)$, and this lies within the $O(1)$ optimal gap.

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